

# First results of the Noether theorem for Hopf-algebra spacetime symmetries\*

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## Abstract

We summarize here the first results obtained using a technique we recently developed for the Noether analysis of Hopf-algebra spacetime symmetries, including the derivation of conserved charges for field theories in noncommutative spacetimes of canonical or  $\kappa$ -Minkowski type.

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## I. INTRODUCTION

In these notes we summarize the first results obtained using a technique for the Noether analysis of Hopf-algebra spacetime symmetries which we developed in Refs. [1, 2, 3, 4]. The mathematics of Hopf algebras is considered as a promising candidate in the search of a formalism compatible with the idea of Planck-scale deformation of spacetime symmetries, possibly in the sense of “doubly-special relativity” [5, 6, 7]. However, progress in this direction has been for a long time stalled by our inability to establish what is the fate of physical/observable aspects of spacetime symmetries in the Hopf-algebra framework. It is legitimate to hope that the Noether charges obtained through our Noether analyses will prove valuable for the debate on these issues.

The Hopf-algebra spacetime symmetries we analyzed [1, 2, 3, 4] are relevant for field theories constructed in canonical noncommutative spacetimes, with the characteristic noncommutativity of coordinates given by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\lambda^2 \Theta^{\mu\nu} \equiv i\theta^{\mu\nu} , \quad (1)$$

or in the so-called  $\kappa$ -Minkowski noncommutative spacetime [8, 9], a Lie-algebra [10] noncommutative spacetime with

$$[\hat{x}_j, \hat{x}_0] = i\lambda \hat{x}_j , \quad [\hat{x}_k, \hat{x}_j] = 0 , \quad (2)$$

where  $\hat{x}_0$  is the time coordinate,  $\hat{x}_j$  are space coordinates ( $j, k \in \{1, 2, 3\}$ ,  $\mu, \nu \in \{0, 1, 2, 3\}$ ),  $\lambda$  is an observer-independent [5] length scale, usually expected to be of the order of the Planck length, and  $\Theta^{\mu\nu}$  ( $\equiv \theta^{\mu\nu}/\lambda^2$ ) is a dimensionless coordinate-independent and observer-independent<sup>1</sup> matrix.

In pursuing the objective of deriving conserved charges from the relevant Hopf-algebra spacetime symmetries we stumbled [1, 2, 3, 4] upon the striking (though, *a posteriori*, obvious) realization that the symmetry-transformation parameters should not commute with the spacetime coordinates. And the form of the relevant commutation relations is such that certain types of “pure transformations” are not allowed. Canonical noncommutative spacetimes admit pure translation transformations, but any transformation involving a Lorentz-sector component must also have a nonvanishing translation component. Similarly in the  $\kappa$ -Minkowski case pure translations and pure space rotations are allowed, but any transformation involving a boost component must also have a nonvanishing space-rotation component.

## II. $\kappa$ -MINKOWSKI NONCOMMUTATIVE SPACETIME

Let us start with the analysis of  $\kappa$ -Minkowski spacetime, following the results we reported in Refs. [1, 2]. For simplicity we shall focus here on the ordering convention<sup>2</sup> such that a generic function of the noncommuting  $\kappa$ -Minkowski coordinates is written as a Fourier sum of “time-to-the-right-ordered” exponentials [19]:

$$\Phi(\hat{x}) = \int d^4k \, \tilde{\Phi}(k) e^{i\vec{k}\cdot\vec{\hat{x}}} e^{-ik_0\hat{x}_0} , \quad (3)$$

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<sup>1</sup> The earliest studies [11] of noncommutativity with a coordinate-independent  $\theta^{\mu\nu}$  actually adopted an even richer formalism, in particular attributing to  $\theta^{\mu\nu}$  some nontrivial algebraic properties [12]. A large literature has been devoted to the simplest picture with a coordinate-independent  $\theta^{\mu\nu}$ , in which  $\theta^{\mu\nu}$  is a (dimensionful) number-valued (observer-dependent) tensor [13, 14, 15], giving rise to a rather familiar type of break down of spacetime symmetries (emergence of a preferred frame). The possibility we considered, the one of a  $\theta^{\mu\nu}$  that is a number-valued observer-independent matrix, was developed more recently, mostly through the works reported in Refs. [16, 17, 18].

<sup>2</sup> As discussed in Refs. [1, 2, 19] one can legitimately adopt other ordering conventions, but for brevity we shall here neglect this possibility.

where the Fourier parameters  $k_\mu$  are commutative and  $\int d^4k$  is an ordinary integral.

We shall consider a (classical) field theory in  $\kappa$ -Minkowski which is invariant under transformations generated by “classical action” translation and space-rotation generators:

$$P_\mu \left( e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0} \right) = k_\mu e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0}, \quad R_j \left( e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0} \right) = i\varepsilon_{jkl} \hat{x}_k k_l e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0}, \quad (4)$$

and boost generators with the rule of action

$$N_j \left( e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0} \right) = \left[ \hat{x}_j \left( \frac{1 - e^{-2\lambda k_0}}{2\lambda} + \frac{\lambda}{2} |\vec{k}|^2 \right) - \hat{x}_0 k_j \right] e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0}, \quad (5)$$

which can be obtained [19] by imposing that the  $N_j$ , together with the  $P_\mu$  and the  $R_j$ , are compatible with the requirements for a Hopf algebra. This turns out to be the  $\kappa$ -Poincaré Hopf algebra (written in the “bicrossproduct basis” [8]).

We shall first argue that the transformations generated by these generators require noncommutative transformation parameters, and then perform a Noether analysis for the corresponding description of symmetry transformations.

In some points of the analysis we shall of course resort to 4D and 3D integration over the  $\kappa$ -Minkowski coordinates, which we shall perform consistently with [1, 2, 19]

$$\int d^4\hat{x} e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0} = \delta^{(4)}(k), \quad (6)$$

and

$$\int d^3\hat{x} e^{i\vec{k}\cdot\vec{x}} e^{-ik_0\hat{x}_0} = \delta^{(3)}(\vec{k}) e^{-ik_0\hat{x}_0}. \quad (7)$$

### A. Noncommutative transformation parameters

In Refs. [1, 2] we sought a description of our transformations on functions  $f(\hat{x})$  of the  $\kappa$ -Minkowski coordinates of the familiar type  $f \rightarrow f + df$ , with

$$df(\hat{x}) = i\gamma^\mu P_\mu f(\hat{x}) + i\tau_j N_j f(\hat{x}) + i\sigma_k R_k f(\hat{x}), \quad (8)$$

and for the transformation parameters  $\gamma^\mu$ ,  $\tau_j$ ,  $\sigma_k$  we insisted that they should act on spacetime coordinates by associative (but not necessarily commutative) multiplication. We further insisted that  $d$  satisfies Leibniz rule,

$$d(f(\hat{x})g(\hat{x})) = (df(\hat{x}))g(\hat{x}) + f(\hat{x})(dg(\hat{x})), \quad (9)$$

and this turns out to be a rather nontrivial requirement, as a result of the fact that from the definitions (4),(5) it follows that [1, 2]

$$P_\mu [f(\hat{x})g(\hat{x})] = [P_\mu f(\hat{x})] g(\hat{x}) + e^{-\lambda P_0(1-\delta_{\mu 0})} f(\hat{x}) [P_\mu g(\hat{x})], \quad (10a)$$

$$N_j [f(\hat{x})g(\hat{x})] = [N_j f(\hat{x})] g(\hat{x}) + e^{-\lambda P_0} f(\hat{x}) [N_j g(\hat{x})] + \lambda \varepsilon_{jkl} [P_k f(\hat{x})] [R_l g(\hat{x})], \quad (10b)$$

$$R_k [f(\hat{x})g(\hat{x})] = [R_k f(\hat{x})] g(\hat{x}) + f(\hat{x}) [R_k g(\hat{x})], \quad (10c)$$

which reflect the structure of the so-called “coproduct” rules of the bicrossproduct basis of the  $\kappa$ -Poincaré Hopf algebra.

Unlike the corresponding transformation parameters for classical Minkowski spacetime, the  $\gamma^\mu$ ,  $\tau_j$  and  $\sigma_k$  must have noncommutative product rules with the coordinates. The commutation

relations of the parameters with the coordinates turn out (as shown<sup>3</sup> in Refs. [1, 2]) to take the following form:

$$\left\{ \begin{array}{l} [\gamma_0, \hat{x}_\mu] = 0 \\ [\gamma_j, \hat{x}_\mu] = i\lambda\gamma_j\delta_\mu^0 \end{array} \right\}, \quad \left\{ \begin{array}{l} [\tau_j, \hat{x}_k] = 0 \\ [\tau_j, \hat{x}_0] = i\lambda\tau_j \end{array} \right\}, \quad \left\{ \begin{array}{l} [\sigma_j, \hat{x}_k] = i\lambda\varepsilon_{jmk}\tau_m \\ [\sigma_j, \hat{x}_0] = 0 \end{array} \right\}. \quad (11)$$

Interestingly these commutators provide an (otherwise unexpected) obstruction for the realization of a pure boost. In fact, according to the commutation relations (11), for  $\tau_m \neq 0$  (at least for some  $m$ ) one necessarily has  $[\sigma_j, \hat{x}_k] \neq 0$  (at least for some  $j, k$  combination), so that at least for some  $j$  one must have  $\sigma_j \neq 0$ : whenever a symmetry transformation has a boost component it must also have a space-rotation component. Clearly no similar obstruction applies to the cases of a pure translation or a pure space rotation: whenever  $\tau_m = 0$  one gets  $[\sigma_j, \hat{x}_\mu] = 0 = [\gamma_\nu, \hat{x}_\mu]$ , so in turn one may also set  $\sigma_j = 0$  and/or  $\gamma_\nu = 0$ .

## B. Noether analysis

The description of symmetry transformations discussed in the previous subsection, encouragingly turns out to allow the derivation of some associated conserved charges. This was verified explicitly in Refs. [1, 2] for the illustrative example of a theory of massless scalar fields  $\Phi(\hat{x})$  governed by the Klein-Gordon-like equation of motion

$$\square_\lambda \Phi(\hat{x}) \equiv \tilde{P}_\mu \tilde{P}^\mu \Phi \equiv \left[ -\left(\frac{2}{\lambda}\right)^2 \sinh^2\left(\frac{\lambda P_0}{2}\right) + e^{\lambda P_0} |\vec{P}|^2 \right] \Phi(\hat{x}) = 0, \quad (12)$$

which is the most studied [9, 19, 22] theory formulated in  $\kappa$ -Minkowski. The operator  $\square_\lambda$  is the “mass Casimir” of the  $\kappa$ -Poincaré Hopf algebra, and we adopted the convenient notation  $\tilde{P}_0 \equiv \left(\frac{2}{\lambda}\right) \sinh(\lambda P_0/2)$ ,  $\tilde{P}_j \equiv e^{\lambda P_0/2} P_j$ .

We of course consider our transformation rules, which for a scalar field take the form

$$\hat{x}'_\mu = \hat{x}_\mu + d\hat{x}_\mu = \hat{x}_\mu + i(\gamma^\nu P_\nu + \tau_j N_j + \sigma_k R_k) \hat{x}_\mu, \quad (13)$$

$$\Phi'(\hat{x}') - \Phi(\hat{x}) = \Phi'(\hat{x}') - \Phi(\hat{x}') + \Phi(\hat{x}') - \Phi(\hat{x}) \simeq \delta\Phi(\hat{x}) + d\Phi(\hat{x}) = 0, \quad (14)$$

with

$$\delta\Phi = -d\Phi \equiv -[i\gamma^\mu P_\mu + i\sigma_j R_j + i\tau_k N_k] \Phi, \quad (15)$$

It is easy to verify [4] that the equation of motion (12) is invariant<sup>4</sup> ( $\delta(\square_\lambda \Phi) = \square_\lambda \delta\Phi = 0$ ) under these transformations.

Our Noether analysis takes as starting point the action

$$S = \frac{1}{2} \int d^4 \hat{x} \Phi(\hat{x}) \square_\lambda \Phi(\hat{x}), \quad (16)$$

from which the equation of motion (12) can be obtained variationally [1].

<sup>3</sup> Also see Ref. [20], which however did not report the correct form of the commutators between transformation parameters, and was eventually revised [21].

<sup>4</sup> Note that the mass Casimir  $\square_\lambda$  commutes with all the generators  $P_\mu, R_j, N_j$  of the Hopf algebra. We also make the natural assumption that the transformation parameters  $\gamma_\mu, \sigma_j, \tau_j$  also commute with  $\square_\lambda$ . In the later section on covariance of the transformation parameters we will present an argument showing that this reasonable assumption does lead to an overall appealing picture.

The result of a variation of the action (16) under our transformation is:

$$\delta S = \frac{1}{2} \int d^4 \hat{x} \tilde{P}^\mu \left\{ \tilde{P}_\mu \left[ \left( e^{\lambda P_0} \Phi \right) \delta \Phi \right] - 2 \left( e^{\lambda P_0} \tilde{P}_\mu \Phi \right) e^{\frac{\lambda}{2} P_0} \delta \Phi \right\}, \quad (17)$$

where we already restricted the analysis to fields that are solutions of the equation of motion (which are the ones whose charges we are interested in), and we used the following property of the operators  $\tilde{P}_\mu$ :

$$\tilde{P}_\mu [f(\hat{x})g(\hat{x})] = \left[ \tilde{P}_\mu f(\hat{x}) \right] \left[ e^{\frac{\lambda}{2} P_0} g(\hat{x}) \right] + \left[ e^{-\frac{\lambda}{2} P_0} f(\hat{x}) \right] \left[ \tilde{P}_\mu g(\hat{x}) \right]. \quad (18)$$

Using the observation [1]

$$\int d^4 \hat{x} e^{\xi P_0} [f(\hat{x})] = \int d^4 \hat{x} f(\hat{x}) \quad \forall \xi, \quad (19)$$

and the fact that from the rules of commutation (11) between transformation parameters and spacetime coordinates it follows that, for a generic function of the coordinates  $f(\hat{x})$ , one has  $f(\hat{x})\gamma_\mu = \gamma_\mu(e^{-\lambda(1-\delta_\mu^0)P_0}f(\hat{x}))$ ,  $f(\hat{x})\tau_j = \tau_j(e^{-\lambda P_0}f(\hat{x}))$  and  $[f(\hat{x}), \sigma_j] = \lambda \varepsilon_{jlk} \tau_l(P_k f(\hat{x}))$ , one can rewrite  $\delta S$  in the following form:

$$\delta S = \int d^4 \hat{x} \left( i\gamma_\nu P_\mu T^{\mu\nu} + i\tau_k P_\mu J_k^\mu + i\sigma_j P_\mu K_j^\mu \right), \quad (20)$$

where:

$$T^{\mu\nu}(\hat{x}) = \frac{1}{2} \left( e^{\lambda(1-\delta_\mu^0)P_0} \tilde{P}^\mu \Phi e^{\frac{\lambda}{2} P_0} P^\nu \Phi - e^{\lambda(1/2-\delta_\mu^0)P_0} \Phi \tilde{P}^\mu P^\nu \Phi \right), \quad (21a)$$

$$\begin{aligned} J_j^\mu(\hat{x}) = & \frac{1}{2} \left( \tilde{P}^\mu \Phi e^{\frac{\lambda}{2} P_0} N_j \Phi - e^{-\frac{\lambda}{2} P_0} \Phi \tilde{P}^\mu N_j \Phi \right) + \\ & + \frac{\lambda}{2} \varepsilon_{jkl} \left( e^{\lambda P_0} \tilde{P}^\mu P_k \Phi e^{\frac{\lambda}{2} P_0} R_l \Phi - e^{\frac{\lambda}{2} P_0} P_k \Phi \tilde{P}^\mu R_l \Phi \right), \end{aligned} \quad (21b)$$

$$K_j^\mu(\hat{x}) = \frac{1}{2} \left( e^{\lambda P_0} \tilde{P}^\mu \Phi e^{\frac{\lambda}{2} P_0} R_j \Phi - e^{\frac{\lambda}{2} P_0} \Phi \tilde{P}^\mu R_j \Phi \right). \quad (21c)$$

And one can explicitly verify [1, 2] that the ten charges obtained by spatial integration of the  $T^{0\mu}(\hat{x})$ ,  $J_j^0(\hat{x})$  and the  $K_j^0(\hat{x})$ ,

$$Q_\mu^P \equiv \int d^3 \hat{x} T_\mu^0(\hat{x}), \quad Q_j^N \equiv \int d^3 \hat{x} J_j^0(\hat{x}), \quad Q_j^R \equiv \int d^3 \hat{x} K_j^0(\hat{x}), \quad (22)$$

are time independent and can be conveniently written, in terms of the Fourier transform  $\tilde{\Phi}(k)$  of a field  $\Phi(\hat{x})$  solution of the equation of motion, as follows:<sup>5</sup>

$$Q_\mu^P = \frac{1}{2} \int d^4 k \frac{\tilde{k}_0}{|\tilde{k}_0|} e^{(2-\delta_\mu^0)\lambda k_0} \tilde{\Phi}(-k_0, -e^{\lambda k_0} \vec{k}) k_\mu \tilde{\Phi}(k) \delta(\tilde{k}^2), \quad (23a)$$

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<sup>5</sup> Here and in the following we will sometimes use the convenient compact notations  $\tilde{k}_0 = 2/\lambda \sinh(\lambda k_0/2)$ ,  $\tilde{k}_j = e^{\frac{\lambda}{2} k_0} k_j$ ,  $\tilde{k}^2 = \tilde{k}_\mu \tilde{k}^\mu$ .

$$Q_j^N = \frac{i}{2} \int d^4k \frac{\tilde{k}_0}{|\tilde{k}_0|} e^{\lambda k_0} \tilde{\Phi}(-k_0, -e^{\lambda k_0} \vec{k}) \delta(\tilde{k}^2) \left\{ k_j \frac{\partial \tilde{\Phi}(k)}{\partial k_0} + \frac{\partial}{\partial k_j} \left[ \left( \frac{1 - e^{-2\lambda k_0}}{2\lambda} - \frac{\lambda}{2} |\vec{k}|^2 \right) \tilde{\Phi}(k) \right] - \lambda k_j \tilde{\Phi}(k) \right\}, \quad (23b)$$

$$Q_j^R = \frac{i}{2} \int d^4k \frac{\tilde{k}_0}{|\tilde{k}_0|} e^{2\lambda k_0} \tilde{\Phi}(-k_0, -e^{\lambda k_0} \vec{k}) \varepsilon_{jlm} k_m \frac{\partial \tilde{\Phi}(k)}{\partial k_l} \delta(\tilde{k}^2). \quad (23c)$$

### III. TRANSLATION TRANSFORMATIONS AND A 5D DIFFERENTIAL CALCULUS

Intriguingly, the Noether analysis reported in the previous section is somehow related to the structure of a known 4D differential calculus for 4D  $\kappa$ -Minkowski: it is easy to verify [1] that the translation-transformation parameters have commutators with the  $\kappa$ -Minkowski coordinates (which we gave in (11)) that exactly reproduce the commutators between elements of the relevant 4D calculus and coordinates. In Ref. [3] (also see Ref. [21]) we followed the same procedure of analysis for a description of translation transformations in 4D  $\kappa$ -Minkowski analogously inspired by another differential calculus, which in particular is a 5D calculus.

The introduction of a differential calculus in  $\kappa$ -Minkowski spacetime is not a trivial matter. For the 4D  $\kappa$ -Minkowski spacetime one finds in the literature a few alternative versions of 4D differential calculus, and even [23] the possibility of a 5D differential calculus defined by the following commutation relations

$$\begin{aligned} [\hat{x}_0, \hat{\gamma}_4] &= i\lambda \hat{\gamma}_0 & [\hat{x}_0, \hat{\gamma}_0] &= i\lambda \hat{\gamma}_4 & [\hat{x}_0, \hat{\gamma}_j] &= 0 \\ [\hat{x}_j, \hat{\gamma}_4] &= [\hat{x}_j, \hat{\gamma}_0] = -i\lambda \hat{\gamma}_j & [\hat{x}_j, \hat{\gamma}_k] &= i\lambda \delta_{jk} (\hat{\gamma}_4 - \hat{\gamma}_0). \end{aligned} \quad (24)$$

where, in light of the intuition that emerged from the analysis reported in the previous section, we denoted the elements of the 5D calculus using the notation  $\hat{\gamma}$  which we intend to adopt for the 5D-calculus-inspired translation-transformation parameters.

As shown in Ref. [3] this choice of transformation parameters suggests a description of the translation-transformation map  $\Phi \rightarrow \Phi + \hat{d}\Phi$  with

$$\hat{d}\Phi = i (\hat{\gamma}^0 \hat{P}_0 + \hat{\gamma}^j \hat{P}_j + \hat{\gamma}^4 \hat{P}_4) \Phi \quad (25)$$

where the operators  $\hat{P}_0, \hat{P}_j, \hat{P}_4$  are simply related to the operators  $P_0, P_j$  considered in the previous section:

$$\begin{aligned} \hat{P}_0 &= \frac{1}{\lambda} (\sinh \lambda P_0 + \frac{\lambda^2}{2} \vec{P}^2 e^{\lambda P_0}) \\ \hat{P}_i &= P_i e^{\lambda P_0} \\ \hat{P}_4 &= \frac{1}{\lambda} (\cosh \lambda P_0 - 1 - \frac{\lambda^2}{2} \vec{P}^2 e^{\lambda P_0}) \end{aligned} \quad (26)$$

Taking into account the coproducts of the operators  $\hat{P}_0, \hat{P}_j, \hat{P}_4$  (which one easily obtains from those of the  $P_0, P_j$ ), and the following useful results on the commutation relations between transformation parameters and time-to-the-right-ordered exponentials

$$\begin{aligned} e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0} \hat{\gamma}_0 &= \left( (\lambda \hat{P}_0 + e^{-\lambda P_0}) \hat{\gamma}_0 + \lambda \hat{P}_i \hat{\gamma}_i + (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \hat{\gamma}_4 \right) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0} \\ e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0} \hat{\gamma}_i &= \left( \lambda e^{-\lambda P_0} \hat{P}_i \hat{\gamma}_0 + \hat{\gamma}_i - \lambda e^{-\lambda P_0} \hat{P}_i \hat{\gamma}_4 \right) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0} \\ e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0} \hat{\gamma}_4 &= \left( \lambda \hat{P}_0 \hat{\gamma}_0 + \lambda \hat{P}_i \hat{\gamma}_i + (\lambda \hat{P}_4 + 1) \hat{\gamma}_4 \right) e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 \hat{x}_0}, \end{aligned}$$

one easily verifies [3] that the differential  $\hat{d}\Phi$  defined in (25) satisfies the Leibniz rule:

$$\hat{d}(\Phi\Psi) = \Phi(\hat{d}\Psi) + (\hat{d}\Phi)\Psi. \quad (27)$$

The specific action we considered in Ref. [3] for the Noether analysis described a free massive scalar field,

$$S[\Phi] = \int d^4\hat{x} \mathcal{L}[\Phi(\hat{x})]$$

$$\mathcal{L}[\Phi(\hat{x})] = \frac{1}{2} (\Phi(\hat{x}) C_\lambda \Phi(\hat{x}) - m^2 \Phi(\hat{x}) \Phi(\hat{x})) , \quad (28)$$

$$(29)$$

so that the Klein-Gordon-like equation of motion takes the form

$$C_\lambda(P_\mu) \Phi \equiv \left[ \left( \frac{2}{\lambda} \sinh \frac{\lambda}{2} P_0 \right)^2 - e^{\lambda P_0} \vec{P}^2 \right] \Phi = m^2 \Phi. \quad (30)$$

We can now analyze the variation of the Lagrangian density under our 5 parameter transformation. Following the same procedure of analysis described in previous section one obtains

$$0 = \delta\mathcal{L} = \frac{1}{2} (\delta\Phi C_\lambda \Phi + \Phi C_\lambda \delta\Phi - m^2 \delta\Phi \Phi - m^2 \Phi \delta\Phi) =$$

$$= -\frac{1}{2} \left\{ e^{\frac{\lambda P_0}{2}} \tilde{P}^0 \left[ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \Phi \delta\Phi - \Phi \frac{e^{-\lambda P_0}}{\lambda} \delta\Phi \right] + \right.$$

$$\left. \hat{P}^i \left[ \Phi e^{-\lambda P_0} \hat{P}_i \delta\Phi - \hat{P}_i \Phi \delta\Phi \right] \right\}, \quad (31)$$

where again we restricted the analysis to fields that are solutions of the equation of motion.

In (31) the transformation parameters  $\hat{\gamma}_A$  appear implicitly through  $\delta\Phi$ . It is convenient to use the formulas (27) to carry all the  $\hat{\gamma}_A$  to the left side of the monomials composing the expression of  $\delta\mathcal{L}$ . This allows to rewrite Eq. (31) in the form

$$\hat{\gamma}^A \left( e^{\frac{\lambda P_0}{2}} \tilde{P}^0 J_{0A} + \hat{P}^i J_{iA} \right) = 0, \quad (32)$$

where

$$J_{00} = \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi + \lambda \hat{P}_i \Phi \hat{P}_i \Phi + \lambda \hat{P}_0 \Phi \hat{P}_4 \Phi \right] + \right.$$

$$\left. - (\lambda \hat{P}_0 + e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi - \lambda \hat{P}_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - \lambda \hat{P}_0 \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \quad (33a)$$

$$J_{0i} = \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \left[ \lambda \hat{P}_i \Phi \hat{P}_0 \Phi + \Phi \hat{P}_i \Phi + \lambda \hat{P}_i \Phi \hat{P}_4 \Phi \right] + \right.$$

$$\left. - \lambda \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi \hat{P}_0 \Phi - \Phi \hat{P}_i \frac{e^{-\lambda P_0}}{\lambda} \Phi - \lambda \hat{P}_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}, \quad (33b)$$

$$\begin{aligned}
J_{04} = & \frac{1}{2} \left\{ \left( \frac{2}{\lambda} + \lambda m^2 - \frac{e^{\lambda P_0}}{\lambda} \right) \cdot \right. \\
& \cdot \left[ (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \hat{P}_0 \Phi - \lambda P_i \Phi \hat{P}_i \Phi + (\lambda \hat{P}_4 + 1) \Phi \hat{P}_4 \Phi \right] + \\
& - (\lambda \hat{P}_4 + 1 - e^{-\lambda P_0}) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_0 \Phi + \\
& \left. + \lambda P_i \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_i \Phi - (\lambda \hat{P}_4 + 1) \Phi \frac{e^{-\lambda P_0}}{\lambda} \hat{P}_4 \Phi \right\}. \tag{33c}
\end{aligned}$$

It is easy to verify that the charges obtained by spatial integration of the  $J_{00}, J_{0i}, J_{04}$  are time independent. For this task it is useful to first notice that our Noether analysis automatically brought in play the operator

$$\hat{\partial}_0 \equiv e^{\frac{\lambda}{2} P_0} \tilde{P}_0 = (\hat{P}_0 + \hat{P}_4) = \frac{e^{\lambda P_0} - 1}{\lambda}, \tag{34}$$

which, unlike [3]  $\hat{P}_0$ , does vanish on any time-independent field.

It is then easy to prove [3] that

$$\hat{\partial}_0 \int d^3 \hat{x} J_{0A} = \int d^3 \hat{x} \hat{\partial}_0 J_{0A} = - \int d^3 \hat{x} \hat{P}^i J_{iA} = 0, \tag{35}$$

from which the time independence of the charges  $\int d^3 \hat{x} J_{0A}$  follows.

Of course, it is not hard to derive an explicit time-independent formula for the charges, which can be most conveniently expressed [3] in terms of the Fourier transform  $\tilde{\Phi}(k)$  of a field  $\Phi(\hat{x})$  solution of the equation of motion:

$$\begin{pmatrix} \hat{Q}_0 \\ \hat{Q}_i \\ \hat{Q}_4 \end{pmatrix} = -\frac{1}{2} \int d^4 k \left| \tilde{\Phi}(k) \right|^2 \begin{pmatrix} \hat{k}_0 \\ \hat{k}_i \\ \hat{k}_4 \end{pmatrix} \frac{(-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2)}{|-2\tilde{k}_0 e^{\frac{\lambda}{2} k_0} + \lambda m^2|} \delta(C_\lambda(k) - m^2), \tag{36}$$

where we used again the notation  $\tilde{k}_0$ , introduced in the preceding section, and we also used the notation

$$\{\hat{k}_0, \hat{k}_i, \hat{k}_4\}|_{k_0, \vec{k}} \equiv \left\{ \frac{1}{\lambda} (\sinh \lambda k_0 + \frac{\lambda^2}{2} \vec{k}^2 e^{\lambda k_0}), k_i e^{\lambda k_0}, \frac{1}{\lambda} (\cosh \lambda k_0 - 1 - \frac{\lambda^2}{2} \vec{k}^2 e^{\lambda k_0}) \right\}.$$

## IV. CANONICAL SPACETIMES

### A. Twisted Hopf symmetry algebra and ordering issues

The type of understanding of the symmetries of  $\kappa$ -Minkowski spacetime that we reported in the previous sections was also achieved, in our paper in Ref. [4], for the symmetries of the canonical noncommutative spacetimes characterized by the noncommutativity given in Eq. (1).

The much studied [16, 17, 18] “twisted” Hopf algebra of (candidate) symmetries of canonical noncommutative spacetime can be obtained by introducing rules of “classical action” [19] for the generators of the symmetry algebra [4]. In fact, observing that the fields one considers in constructing theories in a canonical noncommutative spacetime can be written in the form [10]:

$$\Phi(\hat{x}) = \int d^4 k \tilde{\Phi}_w(k) e^{ik\hat{x}} \tag{37}$$



by introducing ordinary (commutative) “Fourier parameters”  $k_\mu$ , we can associate to any given function  $\Phi(\hat{x})$  a “Fourier transform”  $\tilde{\Phi}_w(k)$ , and it is customary to take this one step further by using this as the basis for an association, codified in a “Weyl map”  $\Omega_w$ , between the noncommutative functions  $\Phi(\hat{x})$  of interest and some auxiliary commutative functions  $\Phi_w^{(comm)}(x)$ :

$$\Phi(\hat{x}) = \Omega_w \left( \Phi_w^{(comm)}(x) \right) \equiv \Omega_w \left( \int d^4 k \tilde{\Phi}_w(k) e^{ikx} \right) = \int d^4 k \tilde{\Phi}_w(k) e^{ik\hat{x}}. \quad (38)$$

It is easy to verify that this definition of the Weyl map  $\Omega_w$  acts on a given commutative function by giving a noncommutative function with full symmetrization (“Weyl ordering”) on the noncommutative spacetime coordinates (*e.g.*,  $\Omega_w(e^{ikx}) = e^{ik\hat{x}}$  and  $\Omega_w(x_1 x_2^2) = \frac{1}{3}(\hat{x}_2^2 \hat{x}_1 + \hat{x}_2 \hat{x}_1 \hat{x}_2 + \hat{x}_1 \hat{x}_2^2)$ ).

It is convenient<sup>6</sup> to use  $\Omega_w$  for our description of the relevant twisted Hopf algebra. This comes about by introducing rules of “classical action” for the generators of translations, space rotations and boosts:<sup>7</sup>

$$P_\mu^{(w)} e^{ik\hat{x}} \equiv P_\mu^{(w)} \Omega_w(e^{ikx}) \equiv \Omega_w(P_\mu e^{ikx}) = \Omega_w(i\partial_\mu e^{ikx}) \quad (39)$$

$$M_{\mu\nu}^{(w)} e^{ik\hat{x}} \equiv M_{\mu\nu}^{(w)} \Omega_w(e^{ikx}) \equiv \Omega_w(M_{\mu\nu} e^{ikx}) = \Omega_w(ix_{[\mu} \partial_{\nu]} e^{ikx}). \quad (40)$$

Here the antisymmetric “Lorentz-sector” matrix of operators  $M_{\mu\nu}$  is composed as usual by the space-rotation generators  $R_i^{(w)} = \frac{1}{2}\varepsilon_{ijk} M_{jk}^{(w)}$  and the boost generators  $N_i^{(w)} = M_{0i}^{(w)}$ . The rules of action codified in (39)-(40) are said to be “classical actions according to the Weyl map  $\Omega_w$ ” since they indeed reproduce the corresponding classical rules of action within the Weyl map.

It is easy to verify that the generators introduced in (39)-(40) satisfy the same commutation relations of the classical Poincaré algebra:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\alpha, M_{\mu\nu}] &= i\eta_{\alpha[\mu} P_{\nu]} \\ [M_{\mu\nu}, M_{\alpha\beta}] &= i(\eta_{\alpha[\nu} M_{\mu]\beta} + \eta_{\beta[\mu} M_{\nu]\alpha}). \end{aligned} \quad (41)$$

However, they close a Hopf (rather than a Lie) algebra because the action of Lorentz-sector generators does not comply with Leibniz rule,

$$\begin{aligned} M_{\mu\nu}^{(w)} (e^{ik\hat{x}} e^{iq\hat{x}}) &= (M_{\mu\nu}^{(w)} e^{ik\hat{x}}) e^{iq\hat{x}} + e^{ik\hat{x}} (M_{\mu\nu}^{(w)} e^{iq\hat{x}}) - \frac{1}{2}\theta^{\alpha\beta} [\eta_{\alpha[\mu} (P_{\nu]}^{(w)} e^{ik\hat{x}}) \\ &\quad (P_\beta^{(w)} e^{iq\hat{x}}) + (P_\alpha^{(w)} e^{ik\hat{x}}) \eta_{\beta[\mu} (P_{\nu]}^{(w)} e^{iq\hat{x}})] , \end{aligned} \quad (42)$$

as one easily verifies using the fact that from (1) it follows that

$$e^{ik\hat{x}} e^{iq\hat{x}} = e^{i(k+q)\hat{x}} e^{-\frac{i}{2}k^\mu \theta_{\mu\nu} q^\nu} \equiv \Omega_w(e^{i(k+q)x} e^{-\frac{i}{2}k^\mu \theta_{\mu\nu} q^\nu}). \quad (43)$$

For the translation generators instead Leibniz rule is satisfied,

$$P_\mu^{(w)} (e^{ik\hat{x}} e^{iq\hat{x}}) = (P_\mu^{(w)} e^{ik\hat{x}}) e^{iq\hat{x}} + e^{ik\hat{x}} (P_\mu^{(w)} e^{iq\hat{x}}), \quad (44)$$

<sup>6</sup> Also in the case of canonical noncommutativity one may consider alternative ordering conventions. The adoption of the Weyl map  $\Omega_w$  essentially corresponds to the choice of a fully symmetrized ordering convention. We have shown in Ref. [4], by analyzing explicitly some alternative choices of ordering, that our results for the charges are independent of this choice of ordering. We shall here for brevity not consider this issue, which readers can find discussed in detail in Ref. [4].

<sup>7</sup> In light of (37) one obtains a fully general rule of action of operators by specifying their action only on the exponentials  $e^{ik\hat{x}}$ . Also note that we adopt a standard compact notation for antisymmetrized indices:  $A_{[\alpha\beta]} \equiv A_{\alpha\beta} - A_{\beta\alpha}$ .

as one could have expected from the form of the commutators (1) which is evidently compatible with classical translation symmetry (while, for observer-independent  $\theta^{\mu\nu}$ , it clearly requires an adaptation of the Lorentz sector.)

In the relevant literature observations of the type reported in (42) and (44) are often described symbolically in the following way

$$\begin{aligned}\Delta P_\mu^{(w)} &= P_\mu^{(w)} \otimes 1 + 1 \otimes P_\mu^{(w)} \\ \Delta M_{\mu\nu}^{(w)} &= M_{\mu\nu}^{(w)} \otimes 1 + 1 \otimes M_{\mu\nu}^{(w)} - \frac{1}{2} \theta^{\alpha\beta} \left[ \eta_{\alpha[\mu} P_{\nu]}^{(w)} \otimes P_\beta^{(w)} + P_\alpha^{(w)} \otimes \eta_{\beta[\mu} P_{\nu]}^{(w)} \right],\end{aligned}\tag{45}$$

where  $\Delta$  is the “coproduct”.

All these results can be expressed in the language of twisted Hopf algebras: the algebra that we have just obtained is the one resulting [4] from the deformation of the classical Poincaré algebra by the twist element:

$$\mathcal{F} = e^{\frac{i}{2} \theta^{\mu\nu} P_\mu^{(w)} \otimes P_\nu^{(w)}}.\tag{46}$$

## B. Noncommutative transformation parameters

In Ref. [4] we provided a description of symmetry transformations in canonical spacetime that follows the same strategy already here described in the previous sections devoted to  $\kappa$ -Minkowski spacetime. We wrote the symmetry-transformation map  $\Phi \rightarrow \Phi + d\Phi$  in terms of the generators  $P_\mu^{(w)}, M_{\mu\nu}^{(w)}$  and of some noncommutative transformation parameters  $\gamma_\mu, \omega_{\mu\nu}$ ,

$$df(\hat{x}) = i \left[ \gamma_{(w)}^\alpha P_\alpha^{(w)} + \omega_{(w)}^{\mu\nu} M_{\mu\nu}^{(w)} \right] f(\hat{x}),\tag{47}$$

and we assumed that the transformation parameters should still act on the spacetime coordinates by simple (associative, but possibly noncommutative) multiplication.

Imposing Leibniz rule on the  $df(\hat{x})$  of Eq. (47) one finds:

$$\begin{aligned}& \left[ [f(\hat{x}), \gamma_{(w)}^\alpha] + \frac{1}{2} \omega_{(w)}^{\mu\nu} (\theta_{[\mu}{}^\alpha \delta_{\nu]}{}^\rho + \theta^\rho{}_{[\mu} \delta_{\nu]}{}^\alpha) (P_\rho^{(w)} f(\hat{x})) \right] P_\alpha^{(w)} g(\hat{x}) + \\ & + [f(\hat{x}), \omega_{(w)}^{\mu\nu}] M_{\mu\nu}^{(w)} g(\hat{x}) = 0,\end{aligned}\tag{48}$$

which amounts (by imposing that the term proportional to  $P_\alpha^{(w)} g(\hat{x})$  and the term proportional to  $M_{\mu\nu}^{(w)} g(\hat{x})$  be separately null) to the following requirements

$$\begin{aligned}\left[ f(\hat{x}), \gamma_{(w)}^\alpha \right] &= -\frac{1}{2} \omega_{(w)}^{\mu\nu} (\theta_{[\mu}{}^\alpha \delta_{\nu]}{}^\rho + \theta^\rho{}_{[\mu} \delta_{\nu]}{}^\alpha) P_\rho^{(w)} f(\hat{x}) \\ \left[ f(\hat{x}), \omega_{(w)}^{\mu\nu} \right] &= 0.\end{aligned}\tag{49}$$

And these requirements imply the following properties of the transformation parameters

$$\left[ \hat{x}^\beta, \gamma_{(w)}^\alpha \right] = -\frac{i}{2} \omega_{(w)}^{\mu\nu} (\theta_{[\mu}{}^\alpha \delta_{\nu]}{}^\beta + \theta^\beta{}_{[\mu} \delta_{\nu]}{}^\alpha)\tag{50}$$

$$\left[ \hat{x}^\beta, \omega_{(w)}^{\mu\nu} \right] = 0.\tag{51}$$

As in the  $\kappa$ -Minkowski case, also here in considering canonical spacetimes we are encountering a restriction on the type of symmetry transformations that are admissible. Specifically in the case of canonical spacetimes there cannot be any pure Lorentz-sector transformation: according to (51,) whenever  $\omega_{(w)}^{\mu\nu} \neq 0$  then also  $\gamma_{(w)}^\mu \neq 0$ . Lorentz-sector transformations are only allowed in combination with some component of translation transformations.

### C. Conserved charges

In Ref. [4] we verified that the description of symmetry transformations provided in the preceding subsection was appropriate for the Noether analysis of theory of scalar massless fields governed by the equation of motion

$$\square\Phi(\hat{x}) \equiv P_{\mu}^{(w)}P_{(w)}^{\mu}\Phi(\hat{x}) = P_{\mu}^{(1)}P_{(1)}^{\mu}\Phi(\hat{x}) = 0. \quad (52)$$

For the laws of transformation of the fields we of course adopted

$$\delta\Phi = -d\Phi = -i \left[ \gamma_{(w)}^{\alpha} P_{\alpha}^{(w)} + \omega_{(w)}^{\mu\nu} M_{\mu\nu}^{(w)} \right] \Phi(\hat{x}), \quad (53)$$

and we considered the following action:

$$S = \frac{1}{2} \int d^4\hat{x} \Phi(\hat{x}) \square\Phi(\hat{x}), \quad (54)$$

which indeed generates the equation of motion (52) and is invariant [4] under the transformation (53)

Focusing on fields that are solutions of the equation of motion, and using the commutation relations (49) between transformation parameters and spacetime coordinates, one finds [4] that the variation of the action can be written in the form

$$\begin{aligned} \delta S &= \frac{1}{2} \int d^4\hat{x} \Phi(\hat{x}) \square\delta\Phi(\hat{x}) = \frac{1}{2} \int d^4\hat{x} P_{\mu}^{(w)} \left[ \Phi(\hat{x}) P_{(w)}^{\mu} \delta\Phi(\hat{x}) - (P_{(w)}^{\mu} \Phi(\hat{x})) \delta\Phi(\hat{x}) \right] \\ &= -i \int d^4\hat{x} \left( \gamma_{\nu}^{(w)} P_{\mu}^{(w)} T^{\mu\nu} + \omega_{(w)}^{\rho\sigma} P_{\mu}^{(w)} J_{\rho\sigma}^{\mu} \right), \end{aligned} \quad (55)$$

where

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2} \left( \Phi(\hat{x}) P_{(w)}^{\mu} P_{(w)}^{\nu} \Phi(\hat{x}) - (P_{(w)}^{\mu} \Phi(\hat{x})) P_{(w)}^{\nu} \Phi(\hat{x}) \right), \\ J_{\rho\sigma}^{\mu} &= \frac{1}{2} \left( \Phi(\hat{x}) P_{(w)}^{\mu} M_{\rho\sigma}^{(w)} \Phi(\hat{x}) - (P_{(w)}^{\mu} \Phi(\hat{x})) M_{\rho\sigma}^{(w)} \Phi(\hat{x}) \right) - \frac{1}{4} (\theta_{[\rho}{}^{\nu} \delta_{\sigma]}{}^{\lambda} + \\ &\quad + \theta^{\lambda}{}_{[\rho} \delta_{\sigma]}{}^{\nu}) \left[ (P_{\lambda}^{(w)} \Phi(\hat{x})) P_{(w)}^{\mu} P_{\nu}^{(w)} \Phi(\hat{x}) - (P_{(w)}^{\mu} P_{\lambda}^{(w)} \Phi(\hat{x})) P_{\nu}^{(w)} \Phi(\hat{x}) \right]. \end{aligned} \quad (56)$$

And we verified [4] explicitly that the charges obtained by spatial integration<sup>8</sup> of the  $T_{\nu}^0, J_{\rho\sigma}^0$ ,

$$Q_{\mu}(\hat{x}_0) = \int d^3\hat{x} T_{\mu}^0, \quad K_{\rho\sigma}(\hat{x}_0) = \int d^3\hat{x} J_{\rho\sigma}^0. \quad (57)$$

are time-independent and can be conveniently written in terms of the Fourier transform  $\tilde{\Phi}_{(w)}(k)$  of a field  $\Phi(\hat{x})$  solution of the equation of motion:

$$Q_{\mu} = \int \frac{d^4q}{4|\vec{q}|} \delta(q^2) \tilde{\Phi}_{(w)}(q) q_{\mu} \left\{ \tilde{\Phi}_{(w)}(-\vec{q}, |\vec{q}|) (q^0 + |\vec{q}|) + \tilde{\Phi}_{(w)}(-\vec{q}, -|\vec{q}|) (q^0 - |\vec{q}|) \right\},$$

$$K_{\rho\sigma} = \int \frac{d^4q}{-4i|\vec{q}|} \delta(q^2) \tilde{\Phi}_{(w)}(q) q_{[\rho} \left\{ (q^0 + |\vec{q}|) \frac{\partial \tilde{\Phi}_{(w)}(-\vec{q}, |\vec{q}|)}{\partial q^{\sigma]} } + (q^0 - |\vec{q}|) \frac{\partial \tilde{\Phi}_{(w)}(-\vec{q}, -|\vec{q}|)}{\partial q^{\sigma]} } \right\}.$$

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<sup>8</sup> We pose  $\int d^3\hat{x} e^{ik^i\hat{x}_i} = \delta^{(3)}(\vec{k})$ .

## V. ASIDE ON COVARIANCE

Our concept of noncommutative transformation parameters has proven very powerful, but it will probably take quite some time to fully appreciate its significance and implications. In this section we want to contemplate the possibility of some rules of action of the symmetry generators on the transformation parameters, just to show that such rules of action can be introduced in a logically-consistent way.

We start from the  $\kappa$ -Minkowski side and with the translation generators. For these we assume<sup>9</sup> that the action on transformation parameters is trivial, just as in the commutative-spacetime limit:

$$P_\mu(\alpha f(\hat{x})) = \alpha P_\mu f(\hat{x}), \quad (58)$$

where  $\alpha$  stands for a generic transformation parameter ( $\alpha = \gamma_\mu, \sigma_j, \tau_k$ ).

In order to develop an analogous intuition for the action of  $\kappa$ -Poincaré space-rotation and boost generators we first observe that, using (4),(5), it is possible to describe these generators in the following way:

$$R_k = \varepsilon_{klm} \hat{x}_l P_m, \quad N_j = \hat{x}_j \bar{P}_0 - \hat{x}_0 P_j, \quad (59)$$

where we introduced  $\bar{P}_0 \equiv (1 - e^{-2\lambda P_0})/2\lambda + \lambda |\vec{P}|^2/2$ . This description of the generators  $R_k$  and  $N_j$  involving exclusively translation generators and spacetime coordinates leads us to assume that the action of  $R_k$  and  $N_j$  on transformation parameters should indeed be derived using (58) and the rules of commutation (11) of the transformation parameters with the spacetime coordinates. It is easy to verify that from this procedure one obtains

$$\begin{aligned} R_k(\alpha f(\hat{x})) &= \varepsilon_{klm} \hat{x}_l \alpha P_m f(\hat{x}) = \alpha R_k f(\hat{x}) + [\hat{x}_l, \alpha] \varepsilon_{klm} P_m f(\hat{x}) \\ N_j(\alpha f(\hat{x})) &= \hat{x}_j \alpha \bar{P}_0 f(\hat{x}) - \hat{x}_0 \alpha P_j f(\hat{x}) = \alpha N_j f(\hat{x}) + [\hat{x}_j, \alpha] \bar{P}_0 f(\hat{x}) - [\hat{x}_0, \alpha] P_j f(\hat{x}), \end{aligned} \quad (60)$$

which can be conveniently reexpressed in the following way:

$$\left\{ \begin{array}{l} [R_j, \gamma_\mu] = 0 \\ [R_j, \tau_k] = 0 \\ [R_j, \sigma_k] = i\lambda(\tau_j P_k - \tau_n P_n \delta_{jk}) \end{array} \right., \quad \left\{ \begin{array}{l} [N_j, \gamma_\mu] = i\lambda \gamma_k \delta_\mu^k P_j \\ [N_j, \tau_k] = i\lambda \tau_k P_j \\ [N_j, \sigma_k] = i\lambda \varepsilon_{jkl} \tau_l \bar{P}_0 \end{array} \right. . \quad (61)$$

And it is also easy to verify that (61) and (58) are fully compatible with the commutation relations (11), and in this sense one might say that those commutation relations are covariant.

A similar analysis is possible in the case of canonical noncommutativity, but it is most easily formulated considering the rules of commutation between transformation parameters and functions of canonical-spacetime coordinates. Following the same reasoning described above in considering the  $\kappa$ -Minkowski case, it is also natural to assume a trivial action of translation generators on the transformation parameters in canonical spacetime:

$$P_\mu^{(w)}(\alpha f(\hat{x})) = \alpha P_\mu^{(w)} f(\hat{x}) \quad (62)$$

where, as before,  $\alpha$  is a generic transformation parameter ( $\alpha = \gamma_\mu, \omega_{\mu\nu}$ ).

For the generators  $M_{\mu\nu}^{(w)}$ , introduced in our description of Lorentz-sector symmetries of canonical spacetimes, we are not aware of any straightforward description in terms of spacetime coordinates

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<sup>9</sup> The translation generators essentially measure the dependence of a quantity on the spacetime coordinates, and even our new transformation parameters remain coordinate independent.

and translation generators. We can still propose rules of action of the  $M_{\mu\nu}^{(w)}$  generators on the transformation parameters by posing

$$M_{\mu\nu}^{(w)}(\alpha f(\hat{x})) = [M_{\mu\nu}^{(w)}, \alpha]f(\hat{x}) + \alpha M_{\mu\nu}^{(w)}f(\hat{x}) , \quad (63)$$

and requiring “covariance” of the commutators (49), *i.e.*

$$M_{\mu\nu}^{(w)}[\alpha, f(\hat{x})] = [M_{\mu\nu}^{(w)}, \alpha]f(\hat{x}) + [\alpha, M_{\mu\nu}^{(w)}f(\hat{x})]. \quad (64)$$

Following this strategy one easily obtains

$$[M_{\mu\nu}^{(w)}, \omega_{\rho\sigma}] = 0, \quad [M_{\mu\nu}^{(w)}, \gamma^\alpha] = \frac{i}{2}\omega^{\rho\sigma} \left( \theta_{[\rho}{}^\alpha \delta_{\sigma]}^\beta - \theta_{[\rho}{}^\beta \delta_{\sigma]}^\alpha \right) \eta_{\beta[\nu} P_{\mu]} . \quad (65)$$

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